

Canonical Functions in Graded Symplectic Geometries and AKSZ Sigma Models

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Abstract

We introduce the notion of canonical functions, which unify many geometric structures in terms of the graded symplectic geometry. From the analysis of canonical functions, we propose generalizations of differential graded symplectic manifolds such as derived QP manifolds, twisted QP manifolds and QP pairs. These explain the nature of kinds of twisted structures in Poisson geometry and physics. Many known and new geometric structures are derived from canonical functions. As an application, AKSZ sigma models and corresponding boundary theories are constructed from canonical functions.

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1 Introduction

Recently the graded symplectic geometry [36][37] has been developed related to new geometric structures and physical theories [29][10]. A QP manifold (a differential graded symplectic manifold) has been introduced from the analysis of the Batalin-Vilkovisky formalism in the gauge theories [32].

In this paper, a canonical function, which is a generalization of graded structures in graded symplectic geometry, is defined and analyzed. This idea is inspired by the canonical transformation on a QP manifold, which is also called twisting [30]. Another motivation comes from the theory of AKSZ topological sigma models [3][9] [31][18] and the topological open membrane theory in physics [28][17]. A canonical function describes the boundary structures of the AKSZ sigma models, which have played key roles in derivation of the deformation quantization from the Poisson sigma model [21][8]. Moreover, it can be viewed as higher analogue of a Poisson function [34][24] and a generalization of the Dirac structure [12] which is a integrability condition of a substructure on the vector bundle.

A canonical function unifies various concepts above separately analyzed until now. This includes the Lie (2-)algebra, the (twisted or quasi) Poisson structure, the (homotopy) Lie algebroid, the (twisted) Courant algebroid, the Nambu-Poisson structure, and so on. Some of them are demonstrated as examples.

The underlying mathematical structures derived from canonical functions are abundant. We prove that any QP manifold \mathcal{M} of degree n is derived from a canonical function on $T^*[n+1]\mathcal{M}$. This analysis leads naturally to new concepts in graded geometry such as the derived and twisted QP manifolds. A general method to twist a QP manifold is given, which is closely connected to the deformation theory. Some examples are explained to illuminate this twisting process. Moreover we find new interesting geometric structures from canonical functions. As one of them, the strong Courant algebroid is proposed. Roughly speaking, it is a Courant algebroid with a bundle map to a Lie algebroid satisfying some compatible conditions.

A consistent AKSZ sigma model [3][9] [31][18] is constructed on a manifold with boundaries using the mathematical structures above. A canonical function guarantees consistency on the bulk and the boundary. Generally some quantum theories on a manifold X in $n+1$ dimensions has the same structure as the corresponding quantum theories on ∂X in n dimensions [5].

We prove this bulk-boundary correspondences of the quantum field theories from derived or twisted QP manifolds in the category of AKSZ sigma models.

The paper is organized as follows. In section 2, a canonical function on a QP manifold is defined. In section 3, their mathematical structures are analyzed. In section 4, AKSZ sigma models with boundaries are constructed from canonical functions. In section 5, the bulk-boundary correspondence of the AKSZ sigma models is clarified and some examples are explained. Section 6 is summary and future outlook. Appendix is a brief explanation of construction of the AKSZ sigma models.

2 Canonical Functions on QP Manifolds

A graded manifold \mathcal{M} on a smooth manifold M is a ringed space with a structure sheaf of a nonnegatively graded commutative algebra over an ordinary smooth manifold M . Grading is compatible with supermanifold grading, that is, a variable of even degree is commutative and one of odd degree is anticommutative. \mathcal{M} is locally isomorphic to $C^\infty(U) \otimes S(V)$, where U is a local chart on M , V is a graded vector space and $S(V)$ is a free graded commutative ring on V . We can refer to [7][35] for the rigorous mathematical definition of objects in supergeometry.

If grading is assumed to be nonnegative, a graded manifold is called a *N-manifold*. Grading is called **degree**.

A N-manifold equipped with a graded symplectic structure ω of degree n is called a *P-manifold* of degree n , (\mathcal{M}, ω) . ω is also called a P-structure. The graded Poisson bracket on $C^\infty(\mathcal{M})$ is defined from the graded symplectic structure ω on \mathcal{M} as $\{f, g\} = (-1)^{|f|+1} i_{X_f} i_{X_g} \omega$, where a Hamiltonian vector field X_f is defined by the equation $i_{X_f} \omega = \delta f$, for $f \in C^\infty(\mathcal{M})$.

Let (\mathcal{M}, ω) be a P-manifold of degree n . A vector field Q of degree $+1$ on \mathcal{M} is called a *Q-structure* if this satisfies $Q^2 = 0$.

Definition 2.1 A triple (\mathcal{M}, ω, Q) is called a **QP-manifold** of degree n and its structure is called a **QP structure**, if ω and Q are compatible, that is, $\mathcal{L}_Q \omega = 0$ [32].

Q is also called a homological vector field. We assume a Hamiltonian $\Theta \in C^\infty(\mathcal{M})$ of Q with

respect to the graded Poisson bracket $\{-, -\}$ satisfying

$$Q = \{\Theta, -\}. \quad (2.1)$$

Θ is of degree $n + 1$. The differential condition, $Q^2 = 0$, implies that Θ is a solution of the **classical master equation**,

$$\{\Theta, \Theta\} = 0. \quad (2.2)$$

Θ satisfying (2.2) is also called a homological function.

Let (\mathcal{M}, ω, Q) be a QP manifold of degree n and suppose that Q is generated by a Hamiltonian function $\Theta \in C^\infty(\mathcal{M})$ of degree $n + 1$. We define an exponential operation e^{δ_α} , $e^{\delta_\alpha}\Theta = \Theta + \{\Theta, \alpha\} + \frac{1}{2}\{\{\Theta, \alpha\}, \alpha\} + \dots$, on the QP manifold \mathcal{M} , where $\alpha \in C^\infty(\mathcal{M})$.

Definition 2.2 Let $(\mathcal{M}, \omega, Q = \{\Theta, -\})$ be a QP manifold of degree n and α be a function of degree n . e^{δ_α} is called a **canonical transformation** if $\Theta' = e^{\delta_\alpha}\Theta$ satisfies $\{\Theta', \Theta'\} = 0$.

Now we introduce the notion of a canonical function.

Definition 2.3 Let $(\mathcal{M}, \omega, Q = \{\Theta, -\})$ be a QP manifold of degree n . A function α of degree n is called a **canonical function** if $e^{\delta_\alpha}\Theta|_{\mathcal{L}} = 0$, where \mathcal{L} is a Lagrangian submanifold with respect to the graded symplectic form ω on \mathcal{M} and $|_{\mathcal{L}}$ is the restriction on \mathcal{L} .

Proposition 2.4 If α is a canonical function on a QP manifold $(\mathcal{M}, \omega, \Theta)$, then e^{δ_α} is a canonical transformation.

Example 2.1 [34] Let \mathcal{M} be a QP manifold of degree 2 defined by $\mathcal{M} = T^*[2](T[1]M \times \mathfrak{g}^*[1])$, where M is a manifold and \mathfrak{g} is a Lie algebra. Local coordinates on M , the fiber of $T[1]M$, and $\mathfrak{g}^*[1]$ are (x^i, q^i, v_a) , respectively. Conjugate coordinates of the fiber of $T^*[2](T[1]M \times \mathfrak{g}^*[1])$ are (ξ_i, p_i, u^a) , respectively. The following function of degree 3:

$$\begin{aligned} \Theta &= \Theta_M + \Theta_C + \Theta_R + \Theta_H \\ &= \xi_i q^i + \frac{1}{2} C_{ab}^c u^a u^b v_c + \frac{1}{3!} R^{abc}(x) v_a v_b v_c + \frac{1}{3!} H_{ijk}(x) q^i q^j q^k, \end{aligned}$$

is a homological function if $\{\Theta_M, \Theta_H\} = 0$ and $\{\Theta_C, \Theta_R\} = 0$. Let H be the 3 form on M defined by $H = \frac{1}{3!} H_{ijk}(x) dx^i \wedge dx^j \wedge dx^k$ and R be a section of $\wedge^3 \mathfrak{g}^*$ defined by $R = \frac{1}{3!} R^{abc}(x) v_a v_b v_c$. Θ is a homological function if and only if H is a closed 3-form on M and R is a closed 3-form associated to the Lie algebra cohomology.

Let $\alpha = \pi + \rho = \frac{1}{2}\pi^{ij}(x)p_ip_j + \rho^j_a(x)u^ap_j$ be a canonical function with respect to the Lagrangian submanifold $\mathcal{L} = T[1]M \times \mathfrak{g}[1] \subset \mathcal{M}$ which is locally expressed by $\mathcal{L} = \{\xi_i = p_i = v_a = 0\}$.

If $\Theta_C = \Theta_R = \Theta_H = 0$ and $\rho = 0$, the canonical function equation $e^{\delta\alpha}\Theta|_{\mathcal{L}} = 0$ is equivalent to $[\pi, \pi]_S = -\{\{\Theta, \pi\}, \pi\}|_{\mathcal{L}} = 0$, which defines a Poisson structure on M . Here $[-, -]_S = -\{\{\Theta, -\}, -\}|_{\mathcal{L}}$ is the Schouten bracket on $\Gamma(\wedge^\bullet TM)$.

If $\Theta_H = 0$, the canonical function equation defines a quasi-Poisson structure, $[\pi, \pi]_S = \wedge^3 \rho^\# R$ [1][2].

If $\rho = 0$, the canonical function equation defines a twisted-Poisson structure, $[\pi, \pi]_S = \wedge^3 \pi^\# H$ [28][20][33].

Generally, a canonical function gives a Lie algebroid structure on $TM \times \mathfrak{g}$ analyzed in [27].

This special kind of canonical function of degree 2 is called a Poisson function in [34][24].

Example 2.2 A **Nambu-Poisson bracket** of order n (≥ 3) on M is a skew symmetric linear map $\{\cdot, \dots, \cdot\} : C^\infty(M)^{\otimes n} \longrightarrow C^\infty(M)$ such that

$$\begin{aligned} (1) \quad & \{f_{\sigma(1)}, f_{\sigma(2)}, \dots, f_{\sigma(n)}\} = (-1)^{\epsilon(\sigma)} \{f_1, f_2, \dots, f_n\}, \\ (2) \quad & \{f_1 g_1, f_2, \dots, f_n\} = f_1 \{g_1, f_2, \dots, f_n\} + g_1 \{f_1, f_2, \dots, f_n\}, \\ (3) \quad & \{f_1, f_2, \dots, f_{n-1}, \{g_1, g_2, \dots, g_n\}\} \\ &= \sum_{k=1}^n \{g_1, \dots, g_k, \{f_1, f_2, \dots, f_{n-1}, g_k\}, g_{k+1}, \dots, g_n\}. \end{aligned}$$

The n -vector field $\pi \in \wedge^n TM$ called a Nambu-Poisson tensor field is defined as $\pi(df_1, df_2, \dots, df_n) = \{f_1, f_2, \dots, f_n\}$.

Let us assume 'decomposability' of the Nambu-Poisson tensor. Then a canonical function for the Nambu-Poisson structure is constructed as follows. Let M be a manifold and consider a N-manifold $\mathcal{M} = T^*[n]T^*[n-1]E[1]$ where $E = \wedge^{n-1}T^*M$. Local coordinates on $T^*[n-1]E[1]$ are denoted by (x^i, v_I, p_i, w^I) of degree $(0, 1, n-1, n-2)$ and conjugate local coordinates of the fiber are (ξ_i, u^I, q^i, z_I) of degree $(n, n-1, 1, 2)$, respectively, where $I = (i_1, i_2, \dots, i_{n-1})$.

A graded symplectic structure of degree n is expressed by $\omega = \delta x^i \wedge \delta \xi_i + \delta v_I \wedge \delta u^I + \delta p_i \wedge \delta q^i + \delta w^I \wedge \delta z_I$.

A Q-structure function Θ is defined as $\Theta = -q^i \xi_i + \frac{1}{(n-1)!} z_I (u^I - q^{i_1} \dots q^{i_{n-1}})$, which trivially satisfies $\{\Theta, \Theta\} = 0$. Θ defines the Dorfman bracket on $TM \oplus \wedge^{n-1}T^*M$ [13] by the

derived bracket $[-, -]_D = \{\{\Theta, -\}, -\}$. We take a function α as

$$\alpha = -\frac{1}{(n-1)!}\pi^{i_1\cdots i_{n-1}i_n}(x)v_{i_1\cdots i_{n-1}}p_{i_n}. \quad (2.3)$$

Note that $\{\alpha, \alpha\} = 0$.

Proposition 2.5 *Let \mathcal{M} , Θ and α be the above ones. Let $\mathcal{L} = T^*[n]E[1]$ be the Lagrangian submanifold of \mathcal{M} . Then α is a canonical function with respect to Θ and \mathcal{L} , i.e., $e^{\delta\alpha}\Theta|_{\mathcal{L}} = 0$ if and only if π is a Nambu-Poisson tensor. [6]*

3 Mathematical Structures derived from Canonical Functions

In this section, we study the underlying mathematical structures derived from canonical functions. Generalizations of QP manifolds such as derived QP manifolds and twisted QP manifolds are obtained. Moreover, a new geometric structure is proposed.

3.1 Derived QP manifolds

Let us take a QP manifold of degree $n+1$ with a canonical function, $(T^*[n+1]\mathcal{M}, \omega, \Theta, \alpha)$. Here \mathcal{M} is a N-manifold and α is the constant on the fiber. Therefore α is regarded as a function on \mathcal{M} . the following derived bracket defines a bracket $\{-, -\}_{\Theta}$ on $C^\infty(\mathcal{M})$:

$$\{f, g\}_{\Theta} = \{\{f, \Theta\}, g\}|_{\mathcal{M}}, \quad (3.4)$$

where f, g are functions on \mathcal{M} , and are viewed as functions on $T^*[n+1]\mathcal{M}$ which are constants along the fiber on the right hand in the equation (3.4). The bracket is graded symmetric on \mathcal{M} because $\{f, g\} = 0$ for $f, g \in C^\infty(\mathcal{M})$, and satisfies the Leibniz rule and the Jacobi identity because of $\{\Theta, \Theta\} = 0$. Throughout this section, we assume that the derived bracket is nondegenerate. Degenerate cases are left to our later work. Thus the bracket $\{-, -\}_{\Theta}$ defines a P-structure on \mathcal{M} . Further, if a canonical function satisfies $\{\alpha, \alpha\}_{\Theta} = 0$, $(\mathcal{M}, \{-, -\}_{\Theta}, \alpha)$ is a QP manifold of degree n . We call this QP manifold the **derived QP manifold**.

A pair of $(T^*[n+1]\mathcal{M}, \omega, \Theta, \alpha)$ and $(\mathcal{M}, \{-, -\}_{\Theta}, \alpha)$ is called a QP pair. We call $(T^*[n+1]\mathcal{M}, \omega, \Theta, \alpha)$ a big QP manifold and $(\mathcal{M}, \{-, -\}_{\Theta}, \alpha)$ a small QP manifold,

Example 3.1 We take $\mathfrak{g}^* = 0$ in Example 2.1 for simplicity. Then a QP manifold is $(T^*[2]T^*[1]M, \omega, \Theta)$ and the graded symplectic structure ω is given by

$$\omega = \delta x^i \wedge \delta \xi_i + \delta p_i \wedge \delta q^i.$$

A Q-structure

$$\Theta = \xi_i q^i + \frac{1}{3!} H_{ijk}(x) q^i q^j q^k, \quad (3.5)$$

is homological if $dH = 0$. A canonical function with respect to the Lagrangian submanifold $\mathcal{L} = T^*[1]M$ is $\alpha = \frac{1}{2} \pi^{ij}(x) p_i p_j$. In local coordinates, the Lagrangian submanifold is given by $\mathcal{L} = \{q^i = \xi_i = 0\}$.

The canonical function equation $e^{\delta\alpha} \Theta|_{\mathcal{L}} = 0$ is equivalent to $\Theta|_{\mathcal{L}'} = 0$, where $\mathcal{L}' \subset \mathcal{M}$ is the Lagrangian submanifold with respect to the reverse canonical transformation of the P-structure ω , which is defined as

$$\begin{aligned} \xi_i &= \{\xi_i, \alpha\} = -\frac{1}{2} \frac{\partial \pi^{jk}}{\partial x^i}(x) p_j p_k, \\ q^i &= \{q^i, \alpha\} = \pi^{ij} p_j. \end{aligned} \quad (3.6)$$

If $H = 0$, the direct computation shows that $\Theta|_{\mathcal{L}'} = 0$ is equal to $\{\alpha, \alpha\}_{\Theta} = 0$ under the derived bracket. $(T^*[1]M, \{-, -\}_{\Theta}, \alpha)$ is a derived QP manifold obtained from $(T^*[2]T^*[1]M, \omega, \Theta, \alpha)$. The derived bracket $\{-, -\}_{\Theta} = \{\{-, \Theta\}, -\}|_{\mathcal{L}}$ is of degree -1 and nothing but the Schouten bracket. The Poisson structure on M is obtained by the derived-derived bracket $\{f(x), g(x)\}_{P.B.} = [f(x), g(x)]_{(\Theta, \alpha)} = \{\{f(x), \alpha\}_{\Theta}, g(x)\}_{\Theta}$, where $f, g \in C^\infty(M)$.

Example 3.2 We consider a N-manifold $\mathcal{M} = T^*[3]T^*[2]E[1]$, where $E \rightarrow M$ is a vector bundle on a manifold M . We take local coordinates (x^i, u^a, p_i) of degree $(0, 1, 2)$ on $T^*[2]E[1]$ and conjugate local coordinates of the fiber (ξ_i, v_a, q^i) of degree $(3, 2, 1)$.

A graded symplectic structure of degree 3 is given by $\omega = \delta x^i \wedge \delta \xi_i + \delta u^a \wedge \delta v_a + \delta p_i \wedge \delta q^i$.

Let us consider the following Q-structure satisfying $\Theta|_{\mathcal{L}} = 0$:

$$\Theta = \xi_i q^i + \frac{1}{2} k^{ab} v_a v_b + \frac{1}{4!} H_{ijkl}(x) q^i q^j q^k q^l,$$

where k^{ab} is a fiber metric on E . Θ satisfies $\{\Theta, \Theta\} = 0$ if and only if $dH = 0$, where $H = \frac{1}{4!} H_{ijkl}(x) dx^i \wedge dx^j \wedge dx^k \wedge dx^l$ is a 4-form on M .

We take the following canonical function of degree 3,

$$\alpha = \rho^i_a(x)p_i u^a + \frac{1}{3!}h_{abc}(x)u^a u^b u^c,$$

with respect to the Lagrangian submanifold $\mathcal{L} = T^*[2]E = \{q^i = \xi_i = v_a = 0\}$. The canonical function equation $e^{\delta\alpha}\Theta|_{\mathcal{L}} = 0$ derives the following identities:

$$k^{ab}\rho^i_a\rho^j_b = 0, \quad (3.7)$$

$$\frac{\partial\rho^i_a}{\partial x^j}\rho^j_b - \frac{\partial\rho^i_b}{\partial x^j}\rho^j_a + k^{cd}\rho^i_c h_{dab} = 0, \quad (3.8)$$

$$\begin{aligned} & \frac{\partial h_{abc}}{\partial x^i}\rho^i_d + k^{ef}h_{abe}h_{fcd} + H_{ijkl}\rho^i_a\rho^j_b\rho^k_c\rho^l_d \\ & + (abcd \text{ completely skewsymmetric}) = 0. \end{aligned} \quad (3.9)$$

If $H = 0$, $e^{\delta\alpha}\Theta|_{\mathcal{L}} = 0$ is equivalent to $\{\alpha, \alpha\}_{\Theta} = 0$. Therefore $(T^*[2]E[1], \{-, -\}_{\Theta}, \alpha)$ is a derived QP manifold of degree 2. The equation (3.7), (3.8) and (3.9) are satisfied if and only if E is a Courant algebroid [26] [29]. The Dorfman bracket of the Courant algebroid is given by the derived-derived bracket on ΓE by

$$[e_1, e_2]_D = \{\{e_1, \Theta\}_{\Theta}, e_2\}_{\Theta}, \quad \forall e_1, e_2 \in \Gamma(E).$$

Example 3.3 Let us consider a N-manifold $\mathcal{M} = T^*[4]T^*[3]E[1]$, where $E \rightarrow M$ is a vector bundle on a manifold M . We take local coordinates (x^i, u^a, w_a, p_i) of degree $(0, 1, 2, 3)$ on $T^*[3]E[1]$ and local coordinates (ξ_i, v_a, z^a, q^i) of degree $(4, 3, 2, 1)$ of the fiber.

A canonical graded symplectic structure of degree 4 is given by

$$\omega = \delta x^i \wedge \delta \xi_i + \delta u^a \wedge \delta v_a + \delta w_a \wedge \delta z^a + \delta p_i \wedge \delta q^i. \quad (3.10)$$

We consider the following Q-structure satisfying $\Theta|_{\mathcal{L}} = 0$:

$$\Theta = \xi_i q^i + v_a z^a + \frac{1}{2}C^a_{ij}(x)v_a q^i q^j + \frac{1}{5!}H_{i_0 i_1 i_2 i_3 i_4}(x)q^{i_0}q^{i_1}q^{i_2}q^{i_3}q^{i_4}.$$

$\{\Theta, \Theta\} = 0$ if and only if $dH = 0$ and $dC = 0$, where $H = \frac{1}{5!}H_{i_0 i_1 i_2 i_3 i_4}(x)dx^{i_0} \wedge dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge dx^{i_4}$ is a 5-form, and $C^a = \frac{1}{2}C^a_{ij}(x)dx^i \wedge dx^j$ is a 2-form taking values on E^* .

Let us consider a following function α of degree 4:

$$\alpha = \rho^i_a(x)u^a p_i + \frac{1}{2}f^a_{bc}(x)w_a u^b u^c + \frac{1}{4!}h_{a_0 a_1 a_2 a_3}(x)u^{a_0}u^{a_1}u^{a_2}u^{a_3}.$$

and the Lagrangian submanifold $\mathcal{L} = \{\xi_i = v_a = z^a = q^i = 0\}$. The canonical function equation $e^{\delta\alpha}\Theta|_{\mathcal{L}} = 0$ derives following conditions for a canonical function α :

$$\frac{\partial \rho^i_a}{\partial x^j} \rho^j_b - \frac{\partial \rho^i_b}{\partial x^j} \rho^j_a + \rho^i_c f^c_{ab} - C^a_{kl} \rho^i_c \rho^k_a \rho^l_b = 0, \quad (3.11)$$

$$\frac{\partial f^a_{bc}}{\partial x^i} \rho^i_d + f^a_{be} f^e_{cd} - C^e_{ij} f^a_{be} \rho^i_c \rho^j_d + (bcd \text{ completely skewsymmetric}) = 0, \quad (3.12)$$

$$\begin{aligned} & \frac{\partial h_{a_0 \dots a_3}}{\partial x^i} \rho^i_{a_4} + h_{ea_2 \dots a_4} f^e_{a_0 a_1} + C^e_{ij} h_{ea_0 \dots a_2} \rho^i_{a_3} \rho^j_{a_4} + H_{j_0 \dots j_4}(x) \rho^{j_0}_{a_0} \rho^{j_1}_{a_1} \rho^{j_2}_{a_2} \rho^{j_3}_{a_3} \rho^{j_4}_{a_4} \\ & + (a_0 a_1 a_2 a_3 a_4 \text{ completely skewsymmetric}) = 0, \end{aligned} \quad (3.13)$$

If $C = H = 0$, the equations above are equivalent to $\{\alpha, \alpha\}_{\Theta} = 0$. Therefore a derived QP manifold of degree 3, $(T^*[3]E[1], \{-, -\}_{\Theta}, \alpha)$, is obtained. The geometric structure of this QP manifold is a Lie algebroid up to homotopy (the splittable H-twisted Lie algebroid) [19][15].

In general, we have the following theorem.

Theorem 3.1 *Any QP manifold is a derived QP manifold.*

Proof Let $(\mathcal{M}, \omega_{\mathcal{M}}, \alpha)$ be a QP manifold. Then we have a canonical graded symplectic structure ω on a cotangent bundle $T^*[n+1]\mathcal{M}$. Generalizing the construction of the Poisson structure on a smooth manifold M from the Schouten bracket and the Poisson bivector on $T^*[1]M$, the canonical Θ is constructed as a degree $n+1$ version of the Poisson bivector such that the Poisson bracket on \mathcal{M} is constructed by the derived bracket [37][23].

Let us take a local coordinate q^i of degree $|q^i|$ on \mathcal{M} such that $\{q^i, q^j\}_{\mathcal{M}} = (\omega_{\mathcal{M}}^{-1})^{ij}(q)$.^c There exists a local coordinate p_i of degree $|p_i|$ on the fiber of $T^*[n+1]\mathcal{M}$ such that $\{q^i, p_j\} = -(-1)^{(|q^i|-n-1)(|p_j|-n-1)}\{p_j, q^i\} = \delta^i_j$ where $\{-, -\}$ is the Poisson bracket defined from the canonical graded symplectic form ω on $T^*[n+1]\mathcal{M}$. If we define $\Theta = -(-1)^{(|q^i|-n-1)(|p_j|-n-1)} \frac{1}{2} (\omega_{\mathcal{M}}^{-1})^{ij}(q) p_i p_j$, $\{\Theta, \Theta\} = 0$, and $\{-, -\}_{\mathcal{M}} = \{\{-, \Theta\}, -\}_{\mathcal{M}}$.

This construction derives the equation $\{\{\Theta, \alpha\}, \alpha\}_{\mathcal{M}} = 0$ and all the terms $\Theta|_{\mathcal{M}}$, $\{\Theta, \alpha\}|_{\mathcal{M}}$, $\{\{\{\Theta, \alpha\}, \alpha\}, \alpha\}|_{\mathcal{M}}$, \dots are zero for the canonical Θ . Therefore we obtain

$$e^{\delta\alpha}\Theta|_{\mathcal{M}} = \Theta|_{\mathcal{M}} + \{\Theta, \alpha\}|_{\mathcal{M}} + \frac{1}{2}\{\{\Theta, \alpha\}, \alpha\}|_{\mathcal{M}} + \frac{1}{3!}\{\{\{\Theta, \alpha\}, \alpha\}, \alpha\}|_{\mathcal{M}} + \dots = 0. \quad (3.14)$$

It says that α is a canonical function for Θ . ■

^c More simply we can also choose a Darboux coordinate on \mathcal{M} .

3.2 Twisted QP Manifolds

In this subsection, we analyze $\{\alpha, \alpha\}_\Theta \neq 0$ case, where α is a canonical function on a QP manifold $(T^*[n+1]\mathcal{M}, \omega, \Theta)$ and $\{-, -\}_\Theta = \{\{-, \Theta\}, -\}|_{\mathcal{M}}$ is the derived bracket on $C^\infty(\mathcal{M})$. In this case, canonical functions contain many kinds of twisted structures, for example, the twisted- or quasi-Poisson structures and the twisted Courant algebroids. Unification of these structures by graded categories leads us to notion of a twisted QP manifold.

Definition 3.2 *Let \mathcal{M} be a P-manifold of degree n with a Poisson bracket $\{-, -\}_{\mathcal{M}}$ and α be a function of degree $n+1$. We call $(\mathcal{M}, \{-, -\}_{\mathcal{M}}, \alpha)$ a **twisted QP manifold** of degree n if there exists a QP manifold $(T^*[n+1]\mathcal{M}, \omega, \Theta)$ such that $\{-, -\}_{\mathcal{M}}$ is given by the derived bracket $\{\{-, \Theta\}, -\}|_{\mathcal{M}}$ with respect to ω and α is a canonical function on $T^*[n+1]\mathcal{M}$. In this case, we say that $(T^*[n+1]\mathcal{M}, \omega, \Theta)$ is a QP realization of $(\mathcal{M}, \{-, -\}_{\mathcal{M}}, \alpha)$.*

A pair of $(T^*[n+1]\mathcal{M}, \omega, \Theta, \alpha)$ and $(\mathcal{M}, \{-, -\}_{\mathcal{M}}, \alpha)$ is called a twisted QP pair.

The following proposition is an immediate consequence of the definition.

Proposition 3.3 *If $(\mathcal{M}, \{-, -\}_{\mathcal{M}}, \alpha)$ is a twisted QP manifold with a QP realization $(T^*[n+1]\mathcal{M}, \omega, \Theta)$, then $\frac{1}{2}\{\alpha, \alpha\}_\Theta = -\Theta|_{\mathcal{M}} - \{\Theta, \alpha\}|_{\mathcal{M}} - \frac{1}{3!}\{\{\{\Theta, \alpha\}, \alpha\}, \alpha\}|_{\mathcal{M}} - \dots$.*

We turn to a typical method to twist a QP manifold. First we take a QP manifold $(\mathcal{M}, \omega_{\mathcal{M}}, \alpha)$. From Theorem 3.1, there exists a QP realization $(T^*[n+1]\mathcal{M}, \omega, \Theta)$, where Θ can be taken as the canonical Q-structure given in the proof of Theorem 3.1. By construction, $\{-, -\}_\Theta = \{\{-, \Theta\}, -\}|_{\mathcal{M}}$ coincides with the Poisson bracket derived from $\omega_{\mathcal{M}}$, and $\{\alpha, \alpha\}_\Theta = \{\{\alpha, \Theta\}, \alpha\}|_{\mathcal{M}} = 0$. That is, $(\mathcal{M}, \{-, -\}_\Theta, \alpha)$ is a derived QP manifold.

Next we consider a deformation of Θ , $\Theta_d = \Theta + \Theta'$ such that $\{\Theta_d, \Theta_d\} = 0$. This condition is equivalent to the Maurer-Cartan equation $Q\Theta' + \frac{1}{2}\{\Theta', \Theta'\} = 0$. Now we concentrate on the deformation such that the derived P-structure on \mathcal{M} is not changed under this deformation, $\omega_{\mathcal{M}, \Theta_d} = \omega_{\mathcal{M}, \Theta}$, i.e., $\{\{-, \Theta'\}, -\}|_{\mathcal{M}}$ is trivial and $\{\{-, \Theta_d\}, -\}|_{\mathcal{M}} = \{\{-, \Theta\}, -\}|_{\mathcal{M}}$. The same function α (Note that α is not a canonical function with respect to Θ now.) is a canonical function with respect to Θ_d if

$$e^{\delta_\alpha} \Theta_d|_{\mathcal{M}} = e^{\delta_\alpha} (\Theta + \Theta')|_{\mathcal{M}} = 0, \quad (3.15)$$

This equation for a deformed Θ_d can break the condition of a derived QP structure, $\{\alpha, \alpha\}_{\Theta_d} = 0$. To see this, we recall that $e^{\delta\alpha}\Theta_d|_{\mathcal{M}} = 0$ derives the equation,

$$\frac{1}{2}\{\alpha, \alpha\}_{\Theta_d} = -(\Theta + \Theta')|_{\mathcal{M}} - \{\Theta + \Theta', \alpha\}|_{\mathcal{M}} - \frac{1}{3!}\{\{\{\Theta + \Theta', \alpha\}, \alpha\}, \alpha\}|_{\mathcal{M}} \cdots.$$

For the canonical Θ , all the terms like $\Theta|_{\mathcal{M}}$ and $\{\{\dots\{\Theta, \alpha\}, \dots, \alpha\}, \alpha\}|_{\mathcal{M}}$ in the right hand are equal to zero. However since some terms in Θ' , for example $\{\{\{\Theta', \alpha\}, \alpha\}, \alpha\}|_{\mathcal{M}}$, can be nonzero, $\{\alpha, \alpha\}_{\Theta_d}$ is not equal to zero in general. Therefore a solution α of the canonical function equation with respect to Θ_d gives us a twisted QP manifold $(\mathcal{M}, \{-, -\}_{\Theta_d}, \alpha)$. Twisting of a QP manifold comes from a deformation of the canonical Θ on $T^*[n+1]\mathcal{M}$. Therefore the following general method to twist a QP manifold is obtained.

Proposition 3.4 *Let $(\mathcal{M}, \omega_{\mathcal{M}}, \alpha)$ be a QP manifold derived from $(T^*[n+1]\mathcal{M}, \omega, \Theta, \alpha)$. Then we obtain a twisted QP manifold $(\mathcal{M}, \omega_{\mathcal{M}}, \alpha)$ if there exists a deformation $\Theta_d = \Theta + \Theta'$, such that*

- (a) $Q\Theta' + \frac{1}{2}\{\Theta', \Theta'\} = 0$;
- (b) $\{\{-, \Theta_d\}, -\}|_{\mathcal{M}} = \{\{-, \Theta\}, -\}|_{\mathcal{M}}$;
- (c) $e^{\delta\alpha}\Theta_d|_{\mathcal{M}} = 0$.

In this case, we say that $(\mathcal{M}, \omega, \alpha)$ is twisted by Θ' .

The equation $Q\Theta' + \frac{1}{2}\{\Theta', \Theta'\} = 0$ is a Maurer-Cartan equation on a QP manifold. It describes deformations of a QP structure. If we consider a one parameter infinitesimal deformation of Q structure Θ , $\Theta_g = \Theta + g\Theta'$. If Θ_g is a Q structure, we have $Q\Theta' = 0$ and $\{\Theta', \Theta'\} = 0$. The first equation is a cohomology condition, while the second one says that Θ' is also a Q structure.

Remark 3.1 Here we have assumed the condition (b) such that the derived P structure on \mathcal{M} is not changed in the twisting process. However, we can consider more general deformations without invariant assumption of the P-structure. Note that the discussion above shows how to twist a known QP manifold, but not all the twisted QP manifolds can be obtained by this way.

We give some illustrative examples of twisted QP manifolds.

Example 3.4 We consider Example 3.1. If H is not equal to zero, Θ is the canonical one plus the H term. From the canonical function equation, a canonical function α satisfies,

$$\frac{1}{2}\{\alpha, \alpha\}_\Theta = -\frac{1}{3!}\{\{\{\Theta, \alpha\}, \alpha\}, \alpha\}|_{T^*[1]M} \neq 0.$$

Therefore a twisted QP manifold $(T^*[1]M, \alpha)$ of degree 1 is obtained. The canonical function equation $e^{\delta\alpha}\Theta|_{\mathcal{L}} = 0$ contains the twisted Poisson structure as already mentioned in Example 2.1. For example, in the case of $H \neq 0$, $\frac{1}{2}\{\alpha, \alpha\}_\Theta = -\frac{1}{3!}\{\{\{H, \alpha\}, \alpha\}, \alpha\}$ derives $[\pi, \pi]_S = \wedge^3 \pi^\# H$. The derived-derived bracket $\{\{-, \alpha\}_\Theta, -\}_\Theta$ gives a twisted Poisson bracket on $C^\infty(M)$.

Example 3.5 (Twisted QP manifolds of degree 2) Let us consider Example 3.2. If $H \neq 0$, Θ is the canonical Q structure plus H . Then a canonical function α satisfies,

$$\frac{1}{2}\{\alpha, \alpha\}_\Theta = -\frac{1}{4!}\{\{\{\{H, \alpha\}, \alpha\}, \alpha\}, \alpha\}|_{T^*[2]E[1]} \neq 0, \quad (3.16)$$

from the canonical function equation. We obtain a twisted QP manifold $T^*[2]E[1]$ of degree 2. The structure given by equations (3.7), (3.8) and (3.9) is the H_4 -twisted Courant algebroid [16], which Leibniz identity of the Dorfman bracket is broken by the 4-form H ^d

$$[e_1, [e_2, e_3]_D]_D - [[e_1, e_2]_D, e_3]_D - [e_2, [e_1, e_3]_D]_D = \wedge^4 \rho^\# H. \quad (3.17)$$

The twisted-Dorfman bracket is given by the derived-derived bracket on ΓE .

Example 3.6 (Twisted QP manifolds of degree 3) A twisted QP manifold of degree 3 is obtained if $H \neq 0$ in Example 3.3 by the similar analysis as Examples above.

Example 3.7 (Twisted QP manifolds of degree n) Let $E \longrightarrow M$ be a vector bundle over a manifold M . We take a N-manifold $(T^*[n]T^*[n-1]E[1], \omega, \Theta)$ with a canonical function α . Local coordinates on $T^*[n-1]E[1]$ are chosen as (x^i, u^a, p_i, w_a) of degree $(0, 1, n-1, n-2)$ and local coordinates of the fiber are (ξ_i, v_a, q^i, z^a) of degree $(n, n-1, 1, 2)$.

The graded symplectic structure of degree n is given by

$$\omega = \delta x^i \wedge \delta \xi_i + \delta u^a \wedge \delta v_a + \delta p_i \wedge \delta q^i + \delta w_a \wedge \delta z^a.$$

^d Note that this is not the Courant algebroid twisted by a closed 3-form H .

We define the Lagrangian submanifold as $\mathcal{L} = T^*[n-1]E[1] = \{\xi_i = v_a = q^i = z^a = 0\}$ and the Q-structure satisfying $\Theta|_{\mathcal{L}} = 0$ as

$$\Theta = \xi_i q^i + v_a z^a + \frac{1}{2} C^a_{ij}(x) v_a q^i q^j + \frac{1}{(n+1)!} H_{i_0 \dots i_n}(x) q^{i_0} \dots q^{i_n}.$$

$\{\Theta, \Theta\} = 0$ derives $dH = 0$ and $dC(x) = 0$, where $H = \frac{1}{(n+1)!} H_{i_0 \dots i_n}(x) dx^{i_0} \wedge \dots \wedge dx^{i_n}$ is a $(n+1)$ -form, and $C^a = \frac{1}{2} C^a_{ij}(x) dx^i \wedge dx^j$ is a 2-form taking values on E^* .

We take the following canonical function α :

$$\alpha = \rho^i_a(x) u^a p_i + \frac{1}{2} f^a_{bc}(x) w_a u^b u^c + \frac{1}{n!} h_{a_1 \dots a_n}(x) u^{a_1} \dots u^{a_n}.$$

$e^\delta_\alpha \Theta|_{\mathcal{L}} = 0$ derives the following equations,

$$\frac{\partial \rho^i_a}{\partial x^j} \rho^j_b - \frac{\partial \rho^i_b}{\partial x^j} \rho^j_a + \rho^i_c f^c_{ab} - C^a_{kl} \rho^i_c \rho^k_a \rho^l_b = 0, \quad (3.18)$$

$$\frac{\partial f^a_{bc}}{\partial x^i} \rho^i_d + f^a_{be} f^e_{cd} - C^e_{ij} f^a_{be} \rho^i_c \rho^j_d + (bcd \text{ completely skewsymmetric}) = 0, \quad (3.19)$$

$$\begin{aligned} & \frac{\partial h_{a_0 \dots a_{n-1}}}{\partial x^i} \rho^i_{a_n} + h_{ea_2 \dots a_n} f^e_{a_0 a_1} + C^e_{ij} h_{ea_0 \dots a_{n-2}} \rho^i_{a_{n-1}} \rho^j_{a_n} + H_{j_0 \dots j_n}(x) \rho^{j_1}_{a_0} \dots \rho^{j_n}_{a_n} \\ & + (a_0 \dots a_n \text{ completely skewsymmetric}) = 0, \end{aligned} \quad (3.20)$$

Then the derived bracket $\{-, -\}_\Theta$ is just the canonical nondegenerate Poisson bracket on $\mathcal{L} = T^*[n-1]E[1]$.

Since $\Gamma(E \otimes \wedge^{n-2} E^*)$ is identified as $C^\infty(T^*[n]E[1])$, we can define a bracket on $\Gamma(E \otimes \wedge^{n-2} E^*)$ by the derived-derived bracket:

$$[-, -]_D = \{\{-, \Theta\}_\Theta, -\}_\Theta|_{\mathcal{L}}. \quad (3.21)$$

If $C = H = 0$, we have $\{\alpha, \alpha\}_\Theta = 0$. From the equations (3.18), (3.19) and (3.20), the derived-derived bracket $[-, -]_D = \{\{-, \alpha\}_\Theta, -\}_\Theta|_{\mathcal{L}}$ is just the higher Dorfman bracket on $E \otimes \wedge^{n-2} E^*$. If C or H is nonzero, $(T^*[n-1]E[1], \{-, -\}_\Theta, \alpha)$ is a twisted QP manifold. If $C = 0$, we obtain the twisted higher Dorfman bracket. The Leibniz identity of the Dorfman bracket is broken by an $n+1$ -form H .

Notion of twisted QP manifolds is applied to physical theories. In next section, we will see that AKSZ sigma models associated to twisted QP manifolds unify AKSZ sigma models and their WZ terms.

3.3 Strong Courant Algebroids

Let E and A be two vector bundles on M . We consider a N-manifold $\mathcal{M} = T^*[3](E[1] \oplus A[1])$. Let us take local coordinates (x^i, u^a, w^p) on M , $E[1]$ and $A[1]$ respectively, and local coordinates (ξ_i, v_a, z_p) of the fiber of $T^*[3]$ of degree $(3, 2, 2)$.

A canonical graded symplectic structure is $\omega = \delta x^i \wedge \delta \xi_i + \delta u^a \wedge \delta v_a + \delta w^p \wedge \delta z_p$.

We define a Q-structure function as

$$\Theta = \frac{1}{2} k^{ab} v_a v_b + \rho^i{}_r(x) \xi_i w^r + \frac{1}{2} C^r{}_{pq}(x) z_r w^p w^q, \quad (3.22)$$

where k^{ab} is a fiber metric on E . $\{\Theta, \Theta\} = 0$ is equivalent to the following identities:

$$\begin{aligned} \rho^i{}_r \frac{\partial \rho^i{}_s}{\partial x^j} - \rho^i{}_s \frac{\partial \rho^i{}_r}{\partial x^j} - \rho^i{}_p C^p{}_{rs} &= 0, \\ -\rho^i{}_p \frac{\partial C^s{}_{qr}}{\partial x^i} + C^s{}_{pt} C^{tqr} + (pqr \text{ cyclic}) &= 0. \end{aligned} \quad (3.23)$$

This condition is satisfied if (A, ρ) is a Lie algebroid, where ρ is a bundle map from A to TM defined by $\rho^i{}_r(x)$.

Let the Lagrangian submanifold $\mathcal{L} \subset \mathcal{M}$ be $\{\xi_i = v_a = w^a = 0\}$, and a canonical function of degree 3 with respect to the Lagrangian submanifold \mathcal{L} be

$$\alpha = \tau^r{}_a(x) z_r u^a + \frac{1}{3!} h_{abc}(x) u^a u^b u^c.$$

The canonical function equation $e^{\delta\alpha} \Theta|_{\mathcal{L}} = 0$ is equivalent to $\Theta|_{\mathcal{L}'} = 0$ on \mathcal{L}' , where \mathcal{L}' is a Lagrangian submanifold with respect to the inverse canonical transformation of the P-structure, $\omega' = -d(e^{-\delta\alpha} \vartheta)$. \mathcal{L}' is defined by

$$\begin{aligned} \xi_i &= \{\xi_i, \alpha\} = -\frac{\partial \tau^r{}_a}{\partial x^i}(x) z_r u^a - \frac{1}{3!} \frac{\partial h_{abc}}{\partial x^i}(x) u^a u^b u^c, \\ v_a &= \{v_a, \alpha\} = -\tau^r{}_a(x) z_r - \frac{1}{2} h_{abc}(x) u^b u^c, \\ w^r &= \{w^r, \alpha\} = \tau^r{}_a(x) u^a. \end{aligned} \quad (3.24)$$

Substituting this equation to $\Theta|_{\mathcal{L}'} = 0$, we obtain the conditions for the canonical function as follows:

$$k^{ab} \tau^r{}_a \tau^s{}_b = 0, \quad (3.25)$$

$$k^{cd} \tau^r{}_c h_{dab} + \rho^i{}_s \tau^s{}_a \frac{\partial \tau^r{}_b}{\partial x^i} - \rho^i{}_s \tau^s{}_b \frac{\partial \tau^r{}_a}{\partial x^i} + C^r{}_{pq} \tau^p{}_a \tau^q{}_b = 0, \quad (3.26)$$

$$\rho^i{}_r \tau^r{}_d \frac{\partial h_{abc}}{\partial x^i} - \frac{1}{2} k^{ef} h_{eab} h_{fcd} + (abcd \text{ complete skewsymmetric}) = 0. \quad (3.27)$$

Now let us derive the geometric structure defined from this canonical function. Let $\tau : E \longrightarrow A$ be a bundle map defined by τ^r_d in local coordinates. Moreover we define the following operations by the derived-derived bracket as follows:

$$[e_1, e_2]_D = -\{\{e_1, \alpha\}_\Theta, e_2\}_\Theta, \quad (3.28)$$

$$\langle e_1, e_2 \rangle = \{e_1, e_2\}_\Theta, \quad (3.29)$$

$$\mathcal{D}(f) = \{\alpha, f\}_\Theta. \quad (3.30)$$

where $\{-, -\}_\Theta := -\{\{-, \Theta\}, -\}|_{\mathcal{L}}$ is the derived bracket, $[-, -]_D$ is a bilinear bracket and $\langle -, - \rangle$ is an inner product on $\Gamma(E)$ and \mathcal{D} is a map from $C^\infty(M)$ to $\Gamma(E)$.

By equations (3.25) (3.26) and (3.27) for $e_1, e_2, e_3 \in \Gamma(E)$ and $\xi_1, \xi_2 \in \Gamma(A^*)$, we have

- (a) $\tau[e_1, e_2]_D = [\tau e_1, \tau e_2]_A$,
- (b) $\langle \tau^*(\xi_1), \tau^*(\xi_2) \rangle = 0$,
- (c) $[e_1, e_1]_D = \mathcal{D}\langle e_1, e_1 \rangle = (\rho \circ \tau)^* d\langle e_1, e_1 \rangle$,
- (d) $[e_1, [e_2, e_3]_D]_D = [[e_1, e_2]_D, e_3]_D + [e_2, [e_1, e_3]_D]_D$,
- (e) $\rho \circ \tau(e_1)\langle e_2, e_3 \rangle = \langle [e_1, e_2]_D, e_3 \rangle + \langle e_2, [e_1, e_3]_D \rangle$.

We call this structure as a strong Courant algebroid.

Definition 3.5 A **strong Courant algebroid** is $(E, \tau, \langle -, - \rangle, [-, -]_D, A, \rho, [-, -]_A)$ satisfying the equation (a) – (e), where E is a vector bundle over a manifold M , $\langle -, - \rangle$ is an inner product on E , $[-, -]_D$ is a bilinear operator, $(A, \rho, [-, -]_A)$ is a Lie algebroid on M and $\tau : E \longrightarrow A$ is a bundle map.

A strong Courant algebroid is a kind of Courant algebroid over a general Lie algebroid. Any Courant algebroid is naturally a strong Courant algebroid.

Proposition 3.6 Let $(E, \tau, \langle -, - \rangle, [-, -]_D, A, \rho, [-, -]_A)$ be a strong Courant algebroid, $(E, \tau \circ \rho, \langle -, - \rangle, [-, -]_D)$ is a Courant algebroid.

From Proposition above, for any strong Courant algebroid, there exists a Courant algebroid associated to it.

Example 3.8 If M is a point, the strong Courant algebroid is a triple $(\mathfrak{g}_1, \mathfrak{g}_2, \tau)$, where \mathfrak{g}_1 is a quadratic Lie algebra, \mathfrak{g}_2 is a Lie algebra and τ is a homomorphism from \mathfrak{g}_1 to \mathfrak{g}_2 such that

$\tau^*(\mathfrak{g}_2^*)$ is an isotropic subspace of \mathfrak{g}_1 . Any Lie bialgebra is an example of the strong Courant algebroid.

Example 3.9 Let A be a Lie algebroid. An inner product on $A \oplus A^*$ is defined by the natural pairing of A and A^* , an anchor map τ is defined by the natural projection from $A \oplus A^*$ to A . It is clear that $(A \oplus A^*, \tau, A, [-, -]_D)$ gives a strong Courant algebroid, where $[-, -]_D$ is the Dorfman bracket given by

$$[X + \xi, Y + \eta]_D = [X, Y]_A + L_X \eta - i_Y d\xi,$$

$X, Y \in \Gamma(A)$, $\xi, \eta \in \Gamma(A^*)$, and L and d are the Lie derivative and the de Rham differential associated to A .

Example 3.10 Let P be a G -principal bundle over M . Since G acts on $TP \oplus T^*P$ naturally, we get a bundle $\frac{TP \oplus T^*P}{G}$ over M by reduction, where the G -invariant sections of $TP \oplus T^*P$ reduce to the sections of the bundle $\frac{TP \oplus T^*P}{G}$.

We define a bundle map τ by the natural projection from $\frac{TP \oplus T^*P}{G}$ to the Atiyah algebroid TP/G . Because of the G -invariance, the canonical Dorfman bracket and the natural pairing on $TP \oplus T^*P$ induce a bracket $[-, -]_D$ and an inner product $\langle -, - \rangle$ on $\Gamma(\frac{TP \oplus T^*P}{G})$. It is easy to confirm that $(\frac{TP \oplus T^*P}{G}, \tau, \langle -, - \rangle, [-, -]_D, TP/G)$ is a strong Courant algebroid.

Following the discussion in subsection 3.3, we can easily generalize the strong Courant algebroid to the twisted version. Most of the concepts appeared in the Courant algebroids can be introduced in parallel to the strong Courant algebroids. For further study about properties of strong Courant algebroids, relations to the A-Connection and morphisms between Lie algebroids will be important [14].

Remark 3.2 In this section, all the examples we have studied are geometric. However, if we study the QP manifold over a point with a canonical function, we can derive kinds of algebraic structures. Twisting of these algebraic structures is also interesting. As a simple example, we can analyze what a underlying algebraic structure is derived from a canonical function on a QP 3 manifold which gives a Lie 2-algebra [4]. For example, see about the Lie 2-algebra derived from a QP manifold and the QP realization of a Lie 2-algebra [11].

4 AKSZ Sigma Models with Boundaries

AKSZ sigma models with boundaries are important applications of canonical functions. Geometric structures of AKSZ sigma models on a base manifold X with boundaries are constructed by a pair of a homological function Θ and a canonical function α .

In $n = 1$ case, this corresponds to a topological open string and derives a deformation quantization formulae [8]. In $n \geq 2$, theories describe topological open membranes [28][17].

4.1 Structures of AKSZ Sigma Models with Boundaries

Let us take a manifold X in $n + 1$ dimensions with nonempty boundaries. Let $(\mathcal{X} = T[1]X, D, \mu)$ be a differential graded manifold over X with a differential D and a compatible measure μ , and \mathcal{M} be a QP manifold of degree n . Then the AKSZ sigma model on the mapping space $\text{Map}(T[1]X, \mathcal{M})$ is constructed by the AKSZ construction. Refer to Appendix for the general theory of the so called AKSZ construction.

For a Q-structure function of the AKSZ sigma model $S = S_0 + S_1$, we denote the integrands by $\widehat{\vartheta}$ and $\widehat{\Theta}$ such that $S_0 = \int_{\mathcal{X}} \mu \widehat{\vartheta}$ and $S_1 = \int_{\mathcal{X}} \mu \widehat{\Theta}$. Boundary conditions on $\partial\mathcal{X}$ must be consistent with the QP structure. If \mathcal{X} has boundaries, the classical master equation becomes $\{S, S\} = \int_{\partial\mathcal{X}} \mu_{\partial\mathcal{X}} (\widehat{\vartheta} + \widehat{\Theta})$, where $\mu_{\partial\mathcal{X}}$ is a boundary measure induced from μ to $\partial\mathcal{X}$. In order to satisfy consistency, $\widehat{\vartheta} + \widehat{\Theta} = 0$ is required on the boundary. Therefore we obtain the following theorem.

Theorem 4.1 $\{S, S\} = 0$ requires $(\vartheta + \Theta)|_{\mathcal{N}} = 0$ on \mathcal{M} , where $\mathcal{N} = \text{Im } \partial\mathcal{X} \subset \mathcal{M}$.

We take Darboux coordinates of superfields with respect to the P-structure ω on $\text{Map}(\mathcal{X}, \mathcal{M})$. 'Superfields' are pullbacks by \mathbf{x}^* of local coordinates on \mathcal{M} . We denote $\mathbf{q}^{a(i)}(\sigma, \theta) \in \Gamma(T[1]X \otimes \mathbf{x}^*(\mathcal{M}_i))$ for $0 \leq i \leq \lfloor n/2 \rfloor$ and $\mathbf{p}_{a(n-i)}(\sigma, \theta) \in \Gamma(T[1]X \otimes \mathbf{x}^*(\mathcal{M}_{n-i}))$ for $\lfloor n/2 \rfloor < i \leq n$, where (σ, θ) are local coordinates on $\mathcal{X} = T[1]X$, \mathcal{M}_i is the degree i subspace of \mathcal{M} , $\mathbf{x} : T[1]X \rightarrow M$ is a map to the base manifold M , and $\lfloor m \rfloor$ is the floor function which gives the largest integer less than or equals to m . The Poisson brackets of superfields are

$$\{\mathbf{q}^{a(i)}(\sigma, \theta), \mathbf{p}_{b(j)}(\sigma', \theta')\} = \delta^i_j \delta^{a(i)}_{b(j)} \delta^{n+1}(\sigma - \sigma') \delta^{n+1}(\theta - \theta'), \quad (4.31)$$

and if $n = \text{even}$ and $i = j = n/2$,

$$\{\mathbf{q}^{a(n/2)}(\sigma, \theta), \mathbf{q}^{b(n/2)}(\sigma', \theta')\} = k^{a(n/2)b(n/2)} \delta^{n+1}(\sigma - \sigma') \delta^{n+1}(\theta - \theta'), \quad (4.32)$$

where $k^{a(n/2)b(n/2)}$ is a metric on $\mathcal{M}_{n/2}$. A differential D on \mathcal{X} is induced from the exterior derivative d on X . This derives a super differential $\mathbf{d} = \theta^\mu \frac{\partial}{\partial \sigma^\mu}$ on $\text{Map}(\mathcal{X}, \mathcal{M})$. If we define $\mathbf{p}_{a(n/2)} = k_{a(n/2)b(n/2)} \mathbf{q}^{a(n/2)}$, S_0 for odd and even n are unified to

$$S_0 = \int_{\mathcal{X}} \mu \widehat{\vartheta} = \int_{\mathcal{X}} \mu \left(\sum_{0 \leq i \leq [n/2]} (-1)^{n+1-i} \mathbf{p}_{a(i)} \mathbf{d} \mathbf{q}^{a(i)} \right). \quad (4.33)$$

In order to derive the equation of motion of superfields from the variational principle, the variation of $S = S_0 + S_1$ must vanish on the boundaries. That is,

$\delta S = \int_{\partial \mathcal{X}} \mu_{\partial \mathcal{X}} \sum_{0 \leq i \leq (n-1)/2} (-1)^{n+1-i} \mathbf{p}_{a(i)} \delta \mathbf{q}^{a(i)} = 0$ is imposed. This equation is satisfied if $\vartheta = 0$ on $\text{Im } \partial \mathcal{X}$. Since $\omega = -\delta \vartheta$, this is to say that $\text{Im } \partial \mathcal{X}$ is a subspace of a Lagrangian submanifold $\mathcal{L} \subset \mathcal{M}$ which is the zero locus of $\vartheta = 0$.

If we take the above physical conditions into account, Theorem 4.1 reduces the simpler geometric condition:

Proposition 4.2 *Let \mathcal{L} be a Lagrangian submanifold of \mathcal{M} , i.e., $\vartheta|_{\mathcal{L}} = 0$. Then $\{S, S\} = 0$ is satisfied if $\Theta|_{\mathcal{L}} = 0$.*

4.2 Canonical Transformations

We carry out a canonical transformation e^{δ_α} by $\alpha \in C^\infty(\mathcal{M})$ of degree n . This changes a target QP manifold to $(\mathcal{M}, \omega, \Theta')$, where a Q-structure is $\Theta' = e^{\delta_\alpha} \Theta$.

For the AKSZ sigma model, though the P-structure does not change, the Q-structure is changed to

$$\begin{aligned} S &= S_0 + S_1 \\ &= \iota_{\widehat{D}} \mu_* \text{ev}^* \vartheta + \mu_* \text{ev}^* e^{\delta_\alpha} \Theta. \end{aligned} \quad (4.34)$$

If $\partial X = \emptyset$, the classical master equation of the theory does not change because of $\{e^{\delta_\alpha} \Theta, e^{\delta_\alpha} \Theta\} = e^{\delta_\alpha} \{\Theta, \Theta\} = 0$. However if $\partial X \neq \emptyset$, α changes the boundary condition of the classical master equation. Applying Proposition 4.1 to the equation (4.34), we obtain the following Proposition [17],

Proposition 4.3 *Let $\partial \mathcal{X} \neq \emptyset$, $(\mathcal{M}, \omega, \Theta)$ be a QP manifold of degree n and $\alpha \in C^\infty(\mathcal{M})$ of degree n be a canonical transformation. Let \mathcal{L} be a Lagrangian submanifold of \mathcal{M} . The*

classical master equation $\{S, S\} = 0$ is satisfied in an AKSZ sigma model (4.34) if $e^{\delta_\alpha} \Theta|_{\mathcal{L}} = 0$, i.e., α is a canonical function.

4.3 From the Canonical Function to the Boundary Term

In the equation (4.34), the change of the Q-structure by the canonical transformation is replaced to the change of the P-structure by the inverse canonical transformation on $\text{Map}(\mathcal{X}, \mathcal{M})$ as

$$\begin{aligned} S' &= e^{-\delta_\alpha} S \\ &= \iota_{\hat{D}} \mu_* \text{ev}^* e^{-\delta_\alpha} \vartheta + \mu_* \text{ev}^* e^{-\delta_\alpha} e^{\delta_\alpha} \Theta. \\ &= \iota_{\hat{D}} \mu_* \text{ev}^* e^{-\delta_\alpha} \vartheta + \mu_* \text{ev}^* \Theta. \end{aligned} \tag{4.35}$$

The AKSZ sigma model and its geometric structure in S' define the same QP structure as the original S on $\text{Map}(\mathcal{X}, \mathcal{M})$.

Let us consider a special case that α satisfies $\{\alpha, \alpha\} = 0$. Then since $e^{-\delta_\alpha} \vartheta = \vartheta - \{\vartheta, \alpha\}$, α generates the so called boundary term, $\mu_{\partial\mathcal{X}*} \text{ev}^* \alpha$ as

$$\begin{aligned} S' &= S_0 - \int_{\mathcal{X}} \mu \mathbf{d} \text{ev}^* \alpha + \int_{\mathcal{X}} \mu \text{ev}^* \Theta \\ &= S_0 - \int_{\partial\mathcal{X}} \mu_{\partial\mathcal{X}} \text{ev}^* \alpha + \int_{\mathcal{X}} \mu \text{ev}^* \Theta \\ &= \iota_{\hat{D}} \mu_* \text{ev}^* \vartheta + \mu_* \text{ev}^* \Theta - \mu_{\partial\mathcal{X}*} \text{ev}^* \alpha. \end{aligned} \tag{4.36}$$

A canonical function introduces a boundary source generated by α , $\int_{\partial\mathcal{X}} \mu_{\partial\mathcal{X}} \text{ev}^* \alpha$, which generally has a nonzero boundary 'charge'. Physically, this sigma model describes topological open membrane [28][17]. Though more general canonical functions generate more complicated boundary terms, these provide physically consistent boundary deformations of the AKSZ sigma models.

5 Bulk-Boundary Correspondence

Generally, a kind of (topological) quantum field theory in $n + 1$ dimensions on X has the same structure as a quantum field theory in n dimensions on ∂X . In the category of AKSZ

sigma models, canonical functions and derived (and twisted) QP manifolds describes this bulk-boundary correspondence.

Let us take a QP manifold of degree $n+1$ with a canonical function, $(T^*[n+1]\mathcal{M}, \omega, \Theta, \alpha)$. First suppose that α satisfies $\{\alpha, \alpha\}_\Theta = \{\{\alpha, \Theta\}, \alpha\}|_{\mathcal{M}} = 0$, i.e., $(\mathcal{M}, \{-, -\}_\Theta, \alpha)$ is a derived QP manifold.

The original QP manifold $(T^*[n+1]\mathcal{M}, \omega, \Theta, \alpha)$ and the derived QP manifold $(\mathcal{M}, \{-, -\}_\Theta, \alpha)$ define two AKSZ sigma models with the same structure. These demonstrate the bulk-boundary correspondence.

Let X be a manifold in $n+1$ dimensions with boundaries. Then $T[1]X$ and $(T^*[n+1]\mathcal{M}, \omega, \Theta, \alpha)$ define an AKSZ sigma models with a P-structure induced by $\omega = \mu_* \text{ev}^* \omega$ and a Q-structure (4.36). Note that if α is a constant on the fiber, $\{\alpha, \alpha\} = 0$.

The AKSZ sigma model on $T[1]\partial X$ with the target QP manifold $(\mathcal{M}, \{-, -\}_\Theta, \alpha)$ is constructed if $\{-, -\}_\Theta$ is nondegenerate and defines the graded symplectic form $\omega_{\mathcal{M}, \Theta}$. The Q-structure is constructed as

$$\begin{aligned} S_{\mathcal{M}} &= S_{\mathcal{M}0} + S_{\mathcal{M}1} \\ &= \iota_{\hat{D}_{\partial X}} \mu_{\partial X*} \text{ev}^* \vartheta_{\mathcal{M}, \Theta} + \mu_{\partial X*} \text{ev}^* \alpha, \end{aligned} \tag{5.37}$$

where $\omega_{\mathcal{M}, \Theta} = -\delta \vartheta_{\mathcal{M}, \Theta}$.

In the physical argument, by integrating out the auxiliary superfields and restricting (4.36) to the Lagrangian submanifold \mathcal{M} , we obtain the Q-structure action (5.37). Therefore (5.37) defined on a derive QP manifold is physically equivalent to (4.36) on the original QP manifold.

Conversely, first we take a QP manifold of degree n , $(\mathcal{M}, \{-, -\}_{\mathcal{M}}, \alpha)$. Since there exist a canonical Q-structure and the graded symplectic form ω on $T^*[n+1]\mathcal{M}$ from Theorem 3.1, we can construct a canonical lift $(T^*[n+1]\mathcal{M}, \omega, \Theta, \alpha)$, where α is a canonical function with respect to Θ . A canonical bulk AKSZ sigma model is constructed from the boundary AKSZ sigma model by this method.

Next we analyze $\{\alpha, \alpha\}_\Theta \neq 0$ case. Let $(\mathcal{M}, \omega_{\mathcal{M}}, \alpha)$ be a twisted QP manifold. By definition, there exists a QP manifold of degree $n+1$ with a canonical function, $(T^*[n+1]\mathcal{M}, \omega, \Theta, \alpha)$.

Let X be a manifold in $n+1$ dimensions with boundaries. $T[1]X$ and $(T^*[n+1]\mathcal{M}, \omega, \Theta, \alpha)$ define an AKSZ sigma model on the bulk $T[1]\partial X$. Generally, it has the Q-structure (4.34).

Next we construct the AKSZ sigma model on $T[1]\partial X$ with the target twisted QP manifold $(\mathcal{M}, \{-, -\}_{\mathcal{M}}, \alpha)$. Since $\{\alpha, \alpha\}_{\mathcal{M}} \neq 0$, in order to construct an equivalent theory on $\partial\mathcal{X}$, a Q-structure function has 'Wess-Zumino terms' to break the master equation.

In the physical argument, by solving the variational equation $\delta_{\mathcal{M}^\perp} S = 0$ on the compliment target space $\mathcal{M}^\perp = T^*[n+1]\mathcal{M}/\mathcal{M}$, the bulk Q-structure reduces the boundary Q-structure represented by only the boundary superfields. We can prove that two AKSZ sigma models constructed on $\text{Map}(T[1]X, T^*[n+1]\mathcal{M})$ and on $\text{Map}(T[1]\partial X, \mathcal{M})$ define the same theories derive from the physical construction.

Let us analyze a previous simple case $\{\alpha, \alpha\} = 0$. (Note that $\{-, -\}$ is the bulk P-structure.) In this case, the correspond bulk Q-structure is expressed by the equation (4.36). Then the equivalent boundary Q-structure of the AKSZ sigma model on $T[1]\partial X$ is obtained as the following boundary theory plus the WZ term:

$$\begin{aligned} S_{\mathcal{M}} &= S_{\mathcal{M}0} + S_{\mathcal{M}1} \\ &= \iota_{\hat{D}_{\partial X}} \mu_{\partial\mathcal{X}*} \text{ev}^* \vartheta_{\Theta, \mathcal{M}} - \mu_{\partial\mathcal{X}*} \text{ev}^* \alpha + \mu_* \text{ev}^* \Theta|_{\delta_{\mathcal{M}^\perp} S=0}. \end{aligned} \quad (5.38)$$

This equation satisfies $\{S_{\mathcal{M}}, S_{\mathcal{M}}\}_{\mathcal{M}} \neq 0$ because of the last WZ term.

Example 5.1 ($n = 2$: Wess-Zumino-Poisson Sigma Models) We consider Example 3.1 as an example of the construction of the AKSZ theories with boundaries in the previous section and this section. The QP manifold is $T^*[2]\mathcal{M} = T^*[2]T[1]M$.

Let us take a manifold X with a boundary in three dimensions whose boundary is a two dimensional manifold ∂X . The AKSZ sigma model is defined on $\text{Map}(T[1]X, T^*[2]T[1]M)$. A P-structure is

$$\omega = \int_{\mathcal{X}} \mu (\delta \mathbf{x}^i \wedge \delta \boldsymbol{\xi}_i + \delta \mathbf{p}_i \wedge \delta \mathbf{q}^i),$$

where the boldface letters are superfields induced from the pullbacks of corresponding local coordinates. The Q-structure function has the following form:

$$S = \int_{\mathcal{X}} \mu \left(-\boldsymbol{\xi}_i d\mathbf{x}^i + \mathbf{q}^i d\mathbf{p}_i + \boldsymbol{\xi}_i \mathbf{q}^i + \frac{1}{3!} H_{ijk}(x) \mathbf{q}^i \mathbf{q}^j \mathbf{q}^k \right), \quad (5.39)$$

if $\alpha = 0$.

In order to obtain the equations of motion from the variational principle, we take the variation of S ,

$$\delta S = \int_{\mathcal{X}} \mu \left(-\delta \boldsymbol{\xi}_i d\mathbf{x}^i - \boldsymbol{\xi}_i d\delta \mathbf{x}^i + \delta \mathbf{q}^i d\mathbf{p}_i + \mathbf{q}^i d\delta \mathbf{p}_i + \delta \left(\boldsymbol{\xi}_i \mathbf{q}^i + \frac{1}{3!} H_{ijk}(x) \mathbf{q}^i \mathbf{q}^j \mathbf{q}^k \right) \right). \quad (5.40)$$

The equations of motion of $\boldsymbol{\xi}$ and \mathbf{q} is obtained by the integration by parts. Then its boundary terms $-\boldsymbol{\xi}_i d\delta\mathbf{x}^i + \mathbf{q}^i d\delta\mathbf{p}_i$, must vanish:

$$\delta S|_{\partial\mathcal{X}} = \int_{\partial\mathcal{X}} \mu_{\partial\mathcal{X}} (-\boldsymbol{\xi}_i \delta\mathbf{x}^i - \mathbf{q}^i \delta\mathbf{p}_i) = 0. \quad (5.41)$$

This derives boundary conditions. The equation (5.41) is satisfied if $\vartheta = 0$ on $\text{Im } \partial\mathcal{X}$. Locally two kind of boundary conditions are possible. $\boldsymbol{\xi}_{//i} = 0$ or $\delta\mathbf{x}_{//}^i = 0$, and $\mathbf{q}_{//}^i = 0$ or $\delta\mathbf{p}_{//i} = 0$, where $//$ is the parallel component to the boundaries ^e.

Now we consider boundary conditions $\boldsymbol{\xi}_{//i} = 0$ and $\mathbf{q}_{//}^i = 0$ as an example. Another consistency condition is that boundary conditions must be consistent with the classical master equation $\{S, S\} = 0$. Direct computation gives

$$\{S, S\} = \int_{\partial\mathcal{X}} \mu_{\partial\mathcal{X}} \left(-\boldsymbol{\xi}_i d\mathbf{x}^i + \mathbf{q}^i d\mathbf{p}_i + \boldsymbol{\xi}_i \mathbf{q}^i + \frac{1}{3!} H_{ijk}(\mathbf{x}) \mathbf{q}^i \mathbf{q}^j \mathbf{q}^k \right), \quad (5.42)$$

Due to the boundary conditions $\boldsymbol{\xi}_{//i} = 0$ and $\mathbf{q}_{//}^i = 0$, the first two terms corresponding to $\widehat{\vartheta}$ of the right hand side vanish on the boundary:

$$\int_{\partial\mathcal{X}} \mu_{\partial\mathcal{X}} \widehat{\vartheta} = \int_{\partial\mathcal{X}} \mu_{\partial\mathcal{X}} (-\boldsymbol{\xi}_i d\mathbf{x}^i + \mathbf{q}^i d\mathbf{p}_i) = 0. \quad (5.43)$$

Therefore the last two terms corresponding $\widehat{\Theta}$ terms in the equation (5.42) must vanish:

$$\int_{\partial\mathcal{X}} \mu_{\partial\mathcal{X}} \widehat{\Theta} = \int_{\partial\mathcal{X}} \mu_{\partial\mathcal{X}} \left(\boldsymbol{\xi}_i \mathbf{q}^i + \frac{1}{3!} H_{ijk}(\mathbf{x}) \mathbf{q}^i \mathbf{q}^j \mathbf{q}^k \right) = 0. \quad (5.44)$$

In this Q-structure, it follows immediately from the boundary conditions $\boldsymbol{\xi}_{//i} = 0$ and $\mathbf{q}_{//}^i = 0$.

This consistency of the boundary conditions is described in terms of a target QP manifolds $T^*[2]\mathcal{M}$. The equation (5.43) is satisfied if $\vartheta|_{\mathcal{M}} = 0$. Under this condition, the equation (5.44) is satisfied if $\text{ev}^*\Theta|_{\partial\mathcal{X}} = 0$. This condition is the pull back of the equation $\Theta|_{\mathcal{M}} = 0$. This corresponds to Proposition 4.2.

Next we introduce α . The Q-structure is modified by introducing a canonical function α for example, $\int_{\partial\mathcal{X}} \mu_{\partial\mathcal{X}} \widehat{\alpha} = \int_{\partial\mathcal{X}} \mu_{\partial\mathcal{X}} \frac{1}{2} \pi^{ij}(\mathbf{x}) \mathbf{p}_i \mathbf{p}_j$. This is the twisted-Poisson structure case. The Q-structure changes to

$$\begin{aligned} S = & \int_{\mathcal{X}} \mu \left(-\boldsymbol{\xi}_i d\mathbf{x}^i + \mathbf{q}^i d\mathbf{p}_i + \boldsymbol{\xi}_i \mathbf{q}^i + \frac{1}{3!} H_{ijk}(\mathbf{x}) \mathbf{q}^i \mathbf{q}^j \mathbf{q}^k \right) \\ & - \int_{\partial\mathcal{X}} \mu_{\partial\mathcal{X}} \frac{1}{2} \pi^{ij}(\mathbf{x}) \mathbf{p}_i \mathbf{p}_j. \end{aligned} \quad (5.45)$$

^eTheir hybrid boundary conditions are also possible.

The boundary term changes boundary conditions. The variation δS changes to

$$\delta S|_{\partial\mathcal{X}} = \int_{\partial\mathcal{X}} \mu_{\partial\mathcal{X}} \left[\left(-\xi_i - \frac{1}{2} \frac{\partial\pi^{ij}(\mathbf{x})}{\partial x^i} \mathbf{p}_j \mathbf{p}_k \right) \delta x^i + (-\mathbf{q}^i - \pi^{ij}(\mathbf{x}) \mathbf{p}_j) \delta \mathbf{p}_i + \cdots \right].$$

Since these terms must vanish, consistent boundary conditions are as follows:

$$\xi_i|_{//} = -\frac{1}{2} \frac{\partial\pi^{jk}}{\partial x^i}(\mathbf{x}) \mathbf{p}_j \mathbf{p}_k|_{//}, \quad \mathbf{q}^i|_{//} = -\pi^{ij}(\mathbf{x}) \mathbf{p}_j|_{//}. \quad (5.46)$$

$\{S, S\} = 0$ requires one another consistency condition, i.e. the integrand of S_1 is zero on the boundary:

$$\left(\xi_i \mathbf{q}^i + \frac{1}{3!} H_{ijk}(\mathbf{x}) \mathbf{q}^i \mathbf{q}^j \mathbf{q}^k \right)|_{//} = 0. \quad (5.47)$$

Similarly, (5.46) and (5.47) can be expressed by the condition on $T^*[2]\mathcal{M} = T^*[2]T[1]M$. Since we use a Q-structure function 4.36, the graded symplectic form is twisted by $e^{-\delta_\alpha}$ in a target QP manifold $T^*[2]\mathcal{M}$. This condition is

$$\xi_i \mathbf{q}^i + \frac{1}{3!} H_{ijk}(x) \mathbf{q}^i \mathbf{q}^j \mathbf{q}^k = 0, \quad (5.48)$$

on the Lagrangian submanifold \mathcal{M}' defined by

$$\xi_i = -\frac{1}{2} \frac{\partial\pi^{jk}}{\partial x^i}(x) p_j p_k, \quad \mathbf{q}^i = -\pi^{ij}(x) p_j. \quad (5.49)$$

This condition corresponds to Proposition 4.3. Substituting (5.49) to (5.48), we obtain the geometric structure on \mathcal{M}' :

$$\begin{aligned} & \xi_i \mathbf{q}^i + \frac{1}{3!} H_{ijk}(x) \mathbf{q}^i \mathbf{q}^j \mathbf{q}^k \\ &= \frac{1}{2} \frac{\partial\pi^{jk}}{\partial x^l}(x) \pi^{li}(x) p_j p_k p_i - \frac{1}{3!} H_{ijk}(x) \pi^{il}(x) \pi^{jm}(x) \pi^{kn}(x) p_l p_m p_n = 0. \end{aligned} \quad (5.50)$$

The equation (5.50) is nothing but the equation of the twisted Poisson structure $[\pi, \pi]_S = \wedge^3 \pi^\# H$.

In the physical argument, the boundary action on $\partial T[1]X$ is obtained by integrating out the superfield ξ_i from the equation (5.45). By integrating out ξ_i , we obtain the Wess-Zumino-Poisson sigma model on \mathcal{M}' :

$$S_{\mathcal{M}} = \int_{\partial\mathcal{X}} \mu_{\partial\mathcal{X}} \left(\mathbf{p}_i d\mathbf{x}^i - \frac{1}{2} f^{ij}(\mathbf{x}) \mathbf{p}_i \mathbf{p}_j \right) + \int_{\mathcal{X}} \mu_{\mathcal{X}} \frac{1}{3!} H_{ijk}(\mathbf{x}) d\mathbf{x}^i d\mathbf{x}^j d\mathbf{x}^k, \quad (5.51)$$

This coincides with the (twisted) AKSZ sigma model constructed from the twisted QP manifold on \mathcal{M} . The P-structure on $\text{Map}(T[1]X, \mathcal{M})$ induced from the P-structure on \mathcal{M} is equivalent to the derived bracket $\{-, -\}_\Theta = \{\{-, \Theta\}, -\}|_{\mathcal{M}}$. The H term breaks the classical master equation with respect to the derived P-structure $\{-, -\}_{\mathcal{M}}$, i.e. $\{S_{\mathcal{M}}, S_{\mathcal{M}}\}_\Theta \neq 0$.

Example 5.2 ($n = 3$: The Twisted Strong Courant Sigma Models) Let us consider the AKSZ sigma model induced from a QP manifold of degree 3 and its canonical function. We take a QP manifold of degree 3, $T^*[3](T^*[2]E[1] \oplus A[1])$, as a generalization of Example 3.2 and Section 3.4, where E and A are two vector bundles on M . Local coordinates on $T^*[2]E[1] \oplus A[1]$ are taken as (x^i, u^a, w^p, p_i) of degree $(0, 1, 1, 2)$, where x^i is a local coordinate on M , u^a is on the fiber of E , w^p is on the fiber of A and p_i is on the fiber of $T^*[2]M$. Conjugate local coordinates of the fiber are (ξ_i, v_a, z_p, q^i) of degree $(3, 2, 2, 1)$. A graded symplectic structure is given by $\omega = \delta x^i \wedge \delta \xi_i + \delta u^a \wedge \delta v_a + \delta p_i \wedge \delta q^i + \delta z_p \wedge \delta w^p$.

We consider the following Q-structure satisfying $\Theta|_{\mathcal{L}} = 0$:

$$\Theta = \xi_i q^i + \frac{1}{2} k^{ab} v_a v_b + \rho^i{}_r(x) \xi_i w^r + \frac{1}{2} C^r{}_{pq}(x) z_r w^p w^q + \frac{1}{4!} H_{ijkl}(x) q^i q^j q^k q^l,$$

where k^{ab} is a fiber metric on E and $H = \frac{1}{4!} H_{ijkl}(x) dx^i \wedge dx^j \wedge dx^k \wedge dx^l$ is a 4-form on M . If A is a Lie algebroid and H is closed, $\{\Theta, \Theta\} = 0$ is satisfied.

We take the Lagrangian submanifold $\mathcal{L} = T^*[2]E[1] \oplus A^*[2] = \{\xi_i = v_a = q^i = w^p = 0\}$, and a function of degree 3,

$$\alpha = \sigma^i{}_a(x) p_i u^a + \tau^r{}_a(x) z_r u^a + \frac{1}{3!} h_{abc}(x) u^a u^b u^c.$$

The canonical function equation $e^{\delta\alpha} \Theta|_{\mathcal{L}} = 0$ determines identities among $\sigma^i{}_a(x)$, $\tau^r{}_a(x)$ and $h_{abc}(x)$ and geometric conditions on \mathcal{L} .

If $w^p = z_p = 0$ and $H = 0$, this reduces to the Courant algebroid. If $w^p = z_p = 0$ and $H \neq 0$, this reduces to the H_4 -twisted Courant algebroid in Example (3.5). If $q^i = p_i = 0$, this reduces to the strong Courant algebroid.

Let us take a manifold X with a boundary in four dimensions, in which boundary is a three dimensional manifold ∂X . The bulk AKSZ sigma model on $T^*[3](T^*[2]E[1] \oplus A[1])$ is constructed by the usual AKSZ construction. The P-structure is

$$\omega = \int_X \mu (\delta \mathbf{x}^i \wedge \delta \boldsymbol{\xi}_i + \delta \mathbf{p}_i \wedge \delta \mathbf{q}^i + \delta \mathbf{u}^a \wedge \delta \mathbf{v}_a + \delta \mathbf{z}_p \wedge \delta \mathbf{w}^p). \quad (5.52)$$

The Q-structure function has the following form:

$$\begin{aligned}
S = & \int_{\mathcal{X}} \mu \left(\xi_i dx^i + q^i dp_i - v_a du^a + w^p dz_p + \xi_i q^i \right. \\
& + \frac{1}{2} k^{ab} v_a v_b + \rho^i_r(x) \xi_i w^r + \frac{1}{2} C^r_{pq}(x) z_r w^p w^q + \frac{1}{4!} H_{ijkl}(x) q^i q^j q^k q^l \Big) \\
& - \int_{\partial \mathcal{X}} \mu_{\partial X} \left(\sigma^i_a(x) p_i u^a + \tau^r_a(x) z_r u^a + \frac{1}{3!} h_{abc}(x) u^a u^b u^c \right). \tag{5.53}
\end{aligned}$$

We can construct the boundary twisted AKSZ sigma model in the method of this section. In this example, generally, the derived Poisson bracket on \mathcal{L} is degenerate. In degenerate cases, it is difficult to express the boundary Q-structure function S . It will be a future problem. However the bulk and boundary theories are physically consistent.

If $A = 0$, a QP manifold is $T^*[3]T^*[2]E[1]$ and the derived Poisson bracket is nondegenerate. The corresponding twisted QP manifold $T^*[2]E[1]$ derives the equivalent boundary topological sigma model with a Wess-Zumino term from a three dimensional manifold ∂X to the target space $T^*[2]E[1]$:

$$\begin{aligned}
S_{\mathcal{M}} = & \int_{\partial \mathcal{X}} \mu_{\partial X} \left(p_i dx^i - \frac{1}{2} k_{ab} u^a du^b - \sigma^i_a(x) p_i u^a - \frac{1}{3!} h_{abc}(x) u^a u^b u^c \right) \\
& + \int_{\mathcal{X}} \mu \frac{1}{4!} H_{ijkl}(x) dx^i dx^j dx^k dx^l. \tag{5.54}
\end{aligned}$$

This (twisted) Q-structure is also obtained by integrating out ξ_i and v_a in the equation (5.53) by physical argument. This is the H_4 -twisted Courant sigma model [16].

6 Summary and Future Outlook

In this paper, we have discussed mathematical and physical applications of canonical functions. They unify many geometric structures in terms of graded geometry and have been used to construct new mathematical structures. Derived, twisted QP manifolds and QP pairs have been proposed as generalizations of QP manifolds. Moreover it has been shown that they unify and explain kinds of twisted structures and the associated AKSZ sigma models with WZ terms and their bulk-boundary correspondences.

However, we do not completely analyze the properties of solutions of canonical function equations and the deformation theory. Deformation theory of QP structure will unify various classical and important structures, for examples, Nijenhuis structures on Lie algebras,

structures in Poisson geometry including the Courant-like algebroids and so on [22]. In order to understand complete structures, we must consider general deformation theory of QP manifolds and canonical functions.

The quantization of the AKSZ sigma models with boundaries is important and considered as next progress. This will not only describe the quantum membrane theories as a physical application but also 'quantize' a wide class of geometric structures.

All the examples derived from canonical functions in this paper are geometric. If we consider a QP manifold and a canonical function over a point, we will derive kinds of algebraic structures. We leave discussion about these questions to future work.

Appendix. AKSZ Construction of Topological Sigma Models

There exists a systematic method to construct a topological sigma model from a differential graded symplectic manifold (a QP manifold), which is called the AKSZ construction [3][9][31]. A resulting sigma model is called an AKSZ sigma model.

Let (\mathcal{X}, D, μ) be a differential graded (dg) manifold \mathcal{X} with a D -invariant nondegenerate measure μ , where D is a differential on \mathcal{X} . Let (\mathcal{M}, ω, Q) be a QP-manifold. ω is a graded symplectic form of degree n and $Q = \{\Theta, -\}$ is a differential on \mathcal{M} . $\text{Map}(\mathcal{X}, \mathcal{M})$ is a space of smooth maps from \mathcal{X} to \mathcal{M} . A QP-structure on $\text{Map}(\mathcal{X}, \mathcal{M})$ is constructed from the above data.

Since $\text{Diff}(\mathcal{X}) \times \text{Diff}(\mathcal{M})$ naturally acts on $\text{Map}(\mathcal{X}, \mathcal{M})$, D and Q induce differentials on $\text{Map}(\mathcal{X}, \mathcal{M})$, \hat{D} and \check{Q} . Explicitly, $\hat{D}(z, f) = \delta f(z)D(z)$ and $\check{Q}(z, f) = Qf(z)$, for $z \in \mathcal{X}$ and $f \in \mathcal{M}^{\mathcal{X}}$.

An *evaluation map* $\text{ev} : \mathcal{X} \times \mathcal{M}^{\mathcal{X}} \longrightarrow \mathcal{M}$ is defined as

$$\text{ev} : (z, f) \longmapsto f(z),$$

where $z \in \mathcal{X}$ and $F \in \mathcal{M}^{\mathcal{X}}$. A *chain map* $\mu_* : \Omega^\bullet(\mathcal{X} \times \mathcal{M}^{\mathcal{X}}) \longrightarrow \Omega^\bullet(\mathcal{M}^{\mathcal{X}})$ is defined as

$$\mu_*\omega(y)(v_1, \dots, v_k) = \int_{\mathcal{X}} \mu(x)\omega(x, y)(v_1, \dots, v_k)$$

where v is a vector field on \mathcal{X} and $\int_{\mathcal{X}} \mu$ is the Berezin integration on \mathcal{X} . The composition $\mu_*\text{ev}^* : \Omega^\bullet(\mathcal{M}) \longrightarrow \Omega^\bullet(\mathcal{M}^{\mathcal{X}})$ is called the *transgression map*.

A **P-structure** (a graded symplectic structure) on $\text{Map}(\mathcal{X}, \mathcal{M})$ is defined as follows:

Definition 6.1 *For a graded symplectic form ω on \mathcal{M} , a graded symplectic form ω on $\text{Map}(\mathcal{X}, \mathcal{M})$ is defined as $\omega := \mu_* \text{ev}^* \omega$.*

ω is nondegenerate and closed because $\mu_* \text{ev}^*$ preserves nondegeneracy and closedness. ω defines a graded Poisson bracket $\{-, -\}$ on $\text{Map}(\mathcal{X}, \mathcal{M})$.

Next a **Q-structure** (a differential) S on $\text{Map}(\mathcal{X}, \mathcal{M})$ is constructed. S consists of two parts $S = S_0 + S_1$. We take a fundamental form ϑ for a P-structure on \mathcal{M} such that $\omega = -\delta\vartheta$ and define $S_0 := \iota_{\hat{D}} \mu_* \text{ev}^* \vartheta$. For a Q-structure Θ on \mathcal{M} , S_1 is constructed as $S_1 := \mu_* \text{ev}^* \Theta$.

It is proved that S is a Q-structure on $\text{Map}(\mathcal{X}, \mathcal{M})$:

$$\{\Theta, \Theta\} = 0 \iff \{S, S\} = 0. \quad (6.55)$$

from definitions of S_0 and S_1 . A homological vector field Q is defined as $Q = \{S, -\}$. $|Q| = 1$ from degree of $\{-, -\}$ and S . The classical master equation shows that Q is a coboundary operator $Q^2 = 0$.

We have the following theorem [3]:

Theorem 6.2 *If \mathcal{X} is a dg manifold with a compatible measure and \mathcal{M} is a QP-manifold, the graded manifold $\text{Map}(\mathcal{X}, \mathcal{M})$ has a QP structure.*

This structure is called an AKSZ sigma model. If $\mathcal{X} = T[1]X$, where X is a manifold in $n + 1$ dimensions, the QP structure on $\text{Map}(\mathcal{X}, \mathcal{M})$ is of degree -1 . A QP structure on $\text{Map}(T[1]X, \mathcal{M})$ is equivalent to the Batalin-Vilkovisky formalism of a topological sigma model.

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